HSE University – Saint-Petersburg

Master program

"Analytics for Business and Economics"

Preparing for the exam in mathematics

1 Linear algebra

1.1 Linear space. Linear dependence of the system of vectors. Basis of a linear space. Scalar product. Orthogonality. Vector length (norm) and distance in Euclidean space.

1.2 Determinant of a square matrix. Calculation of determinants. Decomposition of the determinant in row and column.

1.3 Elementary matrix transformations. Transposition. Matrix rank. Inverse matrix.

1.4 Systems of linear equations. Cramer's method. Gauss method. The maximum (that is, containing the largest possible number of elements) set of linearly independent solutions of a system.

1.5 Eigenvalues and eigenvectors of square matrices. Properties of eigenvalues and eigenvectors of some special classes of matrices: symmetric matrices, non-negative matrices (Frobenius-Perron theorem), orthogonal projectors.

1.6 Square shapes. Matrix of quadratic form. The condition of positive (negative) definiteness is quadratic. Sylvester's criterion.

1.7 Matrix norm. Matrix norms subject to vector norms in $\mathbb{R}^n$. The concept of contracting linear mapping. Matrix inversion of $\boldsymbol{I} - \boldsymbol{A}$. Neumann series.

1.8 Elements of analytical geometry. Equation of a straight line on a plane. The equation of a straight line and a plane in three-dimensional space. Plane curves of the second order: the equation of an ellipse, parabola, hyperbola. Reduction of a matrix to a diagonal form. Finding the principal axes of an ellipse and a hyperbola.

2 Mathematical analysis

2.1 Sets. Operations on sets. Numeric sets. Sets in $\mathbb{R}^n$. Set correspondence. Countable and uncountable sets.

2.2 Numerical sequences and limits. Properties of convergent sequences. Signs of the existence of a limit. Calculation of limits.

2.3 Functions of one variable. Derivatives. Analysis and construction of a graph of a function.

2.4 Unconditional extremum of a function of one variable. Weierstrass' theorem on the largest and smallest value of a function. First order condition for internal extremum. Problems with a parameter for finding internal and boundary extrema of a function of one variable.

2.5 Functions of many variables. Private derivatives. Full differential. Function gradient. Directional derivative. Hessian matrix.

2.6 Unconditional extremum of a function of several variables. Necessary and sufficient conditions for an internal extremum of a function of several variables. Problems with a parameter for finding extrema of a function of several variables.

2.7 Convex sets. Convex and concave functions. Jensen's inequality. Conditions for convexity and concavity of differentiable and twice differentiable functions.

2.8 Optimization in the presence of restrictions. Method of Lagrange multipliers. Hessian. The second order condition for the internal extremum. Corner solutions. The Kuhn-Tucker theorem. Problems with a parameter for finding internal and border extrema of a function of several variables.

2.9 Indefinite integral and its calculus. Definite integral. Improper integrals. Multiple integrals and their calculus.

2.10 The concept of a series and its convergence. Properties of convergent series. Convergence criteria for positive series. Variable rows. Functional series. Uniform convergence of a functional series.

2.11 Power series. Radius of convergence of a power series. Integration and differentiation of power series. Taylor series. Expansion of a function in a Taylor series with a remainder term in the Peano form.

3 Ordinary differential equations

3.1 Ordinary differential equations (ODE) of the first order. The concept of general and particular solutions to ODEs. Integral curves. Cauchy problem.

3.2 Equation in total differentials. Change of variables method. Integrating factor. Bernoulli and Riccati equation.

3.3 Linear differential equations of the 1st order. Variation of constans method. Linear differential equations of the nth order.

3.4 Homogeneous linear differential equations with constant coefficients. Characteristic equation. Stability of the solution according to Lyapunov.

3.5 Not homogeneous linear differential equations with constant coefficients and with the right side in the form of a quasi-polynomial.

3.6 System of linear differential equations. The concept of stability of solutions of a dynamical system. Stability of solutions according to Lyapunov. Asymptotic stability.

3.7 Autonomous ODEs. Elements of qualitative analysis. Phase diagrams and their use for establishing local stability of stationary solutions.

3.8 Initial information about difference equations and systems of difference equations. Nonlinear mapping iterations, contraction mapping theorem.

4 Probability theory

4.1 Random events and their probability. The concept of independence. Conditional Probability.

4.2 Random variables and probability distribution laws. Discrete and continuous distributions. Cumulative distribution function. Density function of a continuous distribution. Examples of discrete distributions: binomial, geometric, Poisson distribution. Examples of continuous distributions: uniform, exponential, power (Pareto), normal. Characteristics of random variables: mathematical expectation, variance, mode, median.

4.3 Distributions related to the normal distribution: lognormal distribution, χ^2distribution, truncated normal distribution, Student's and Fisher's distributions. Local and integral theorems of Moivre-Laplace.

4.4 Joint distribution of two random variables. Conditional distribution. Conditional mathematical expectation, conditional variance. Correlation coefficient. Linear regression. Two-dimensional normal distribution and its characteristics. Bayes formula.

5 Mathematical statistics

5.1 General population and sample. Sample distribution and sample characteristics (mean, variance, covariance, correlation coefficient).

5.2 Statistical analysis. Point estimates. Linearity, unbiasedness, efficiency and consistency of estimates. Interval estimates, confidence interval. The Method of Moments and the Maximum Likelihood Method for Point Estimation of Distribution Parameters.

5.3 Statistical inference and testing of statistical hypotheses. Confidence level and significance testing. Examples of testing hypotheses: hypothesis about the equality of means, hypothesis about the equality of the mean to a given value.

6 Discrete mathematics

6.1 Binary relations and their properties (reflexivity, transitivity, symmetry, completeness). Equivalence relation. Order relation.

6.2 Basic information about graphs. Directed and undirected graphs. Connectedness and strong connectedness of graphs. Connectivity components. Graph adjacency matrix.

6.3 Bipartite graphs. Matching. The maximum matching problem.

Literature

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Master program

"Data Analytics for Business and Economics"

Demo exam in higher mathematics

Solve the following 10 problems from the suggestions below. The duration of the exam is 90 minutes.

The examination paper is evaluated on a 100-point scale, the maximum weight of each task is 12 points.

During the exam, applicants are allowed to use only writing accessories.

Task 1. Find the function value set $g(x, y, z) \equiv 5x^2 + 2y^2 - z^2$ for the set

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}.$$

Task 2. Given a function

$$F(x, y) = \frac{x + y}{1 + x + y} - x \ln x - y \ln y, \tag{1}$$

defined for all vectors (x, y) with positive coordinates. Write the equation of the tangent plane to the graph of the function given by the equation (1) at the point $(x, y) = (1, 3)$.

Problem 3. Given the optimization problem:

$$\min_{(x,y)\in\mathbb{R}^2} px + qy \quad \text{subject to: } x \geq 0,\ y \geq 0,\ \alpha \ln(x) + (1 - \alpha)\ln(y) \geq 0, \tag{2}$$

where p, q and α are positive real parameters, and $\alpha < 1$. Find expressions for the optimum point and the optimal value of the objective function in the (2) problem in terms of p, q and α parameters.

Problem 4. Given the optimization problem:

$$\max_{(x,y)\in\mathbb{R}^2} x + \ln y, \quad \text{subject to:} \quad x > 0,\ y \geq 0,\ x + y \leq w, \tag{3}$$

where w is a positive real parameter.

1) For what values of the parameter w is the admissible set of the problem (3) non-empty?

2) For what values of the parameter w does the problem (3) have an internal solution?

Problem 5. Testing is carried out to detect a certain disease. The test gives a "true positive"result (that is, a sick person gets a positive result) for 94% of patients. The test gives a "true negative"result (that is, a healthy person gets a negative result) for 99% of healthy people. It is known that 152 out of every 100,000 people in the population are sick. What is the probability that a person is sick if they get a positive test result?

Problem 6. The random variables X and Y are independent, and each of them is distributed according to a standard normal distribution, and the random variables Z and W are given as follows:

$$Z \equiv X - Y, \quad W \equiv e^X.$$

1) Find the joint distribution density of the random vector (X, Z).

2) Find the conditional expectations $\mathbb{E}(X|Z)$ and $\mathbb{E}(X|W)$.

Problem 7. There are two random variables, $\tilde{x}$ and $\tilde{y}$, with joint density $f(x, y)$ given by the following table:

$f(x,y)$	$\tilde{y}=10$	$\tilde{y}=20$	$\tilde{y}=30$
$\tilde{x}=1$	0.04	0	0.20
$\tilde{x}=2$	0.07	0	0.18
$\tilde{x}=3$	0.02	0.11	0.07
$\tilde{x}=4$	0.01	0.12	0.18

1. Construct a table showing the distribution function $F(x; y)$ 1.
2. Find the one-dimensional distributions of $F_{\tilde{x}}(x)$ and $F_{\tilde{y}}(y)$.
3. Find the distribution densities $f_{\tilde{x}}(x)$ and $f_{\tilde{y}}(y)$.
4. Find the conditional density $f(x|\tilde{y} = 20)$.
5. Find the average value of $\tilde{y}$.
6. Find the average value of $\tilde{x}$, assuming that $\tilde{y} = 20$.
7. Are $\tilde{x}$ and $\tilde{y}$ independent variables?

Problem 8. Find eigenvalues and eigenvectors of matrices

$$\mathbf{A} \equiv \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \qquad \mathbf{B} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where ρ is a real parameter.

For what values of ρ do the eigenvectors of the matrix $\mathbf{A}$ form a basis in the plane $\mathbb{R}^2_+$?

Prove (or disprove) that the eigenvectors of the matrix $\mathbf{B}$ form a basis in the plane $\mathbb{R}^2_+$.

Problem 9. For a given differential equation, draw its integral curves:

$$ydx + 2xdy = 0.$$

Problem 10. Given a differential equation, where $y = y(x)$:

$$y' - \frac{2}{x+1}y = e^x (x+1)^2.$$

Find its general solution and solve the Cauchy problem for the initial condition: $y(0) = 1$.