# Frugality, Solidarity, and Context-Dependent Fairness (and Efficiency)

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# Motivation

**When public utilities exhibit decreasing returns to scale, frugality should be rewarded.**

A cost-sharing rule should:

- not incentivize consumption
  - negative cost shares and free lunches should be avoided
- reward consumption reductions
  - cost savings should not be used to subsidize consumption increases by others, especially not large users.

**Average cost pricing, two agents**

- $q_1 = q_2 = Q/2 \quad \Longrightarrow \quad x_1 = x_2 = C(Q)/2$
- Consider a transfer of $\delta > 0$:

$$q_1' = Q/2 - \delta \quad \text{and} \quad q_2' = Q/2 + \delta$$

- Cost shares become:

$$x_1' = \frac{C(Q)}{Q}(Q/2 - \delta) \qquad \text{and} \qquad x_2' = \frac{C(Q)}{Q}(Q/2 + \delta).$$

# A counterexample (cont.)

**Compared to the situation where both consume the average amount,** $Q/2$:

- Agent 2 pays an extra $x_2' - x_2 = \delta C(Q)/Q$
- Agent 1 gets (an equivalent) rebate $x_1 - x_1' = \delta C(Q)/Q$
- If costs are convex, the average cost is less than the marginal cost:
$$\frac{C(Q)}{Q} < C'(Q).$$
- The rebate to Agent 1 is less than the cost savings she generates. The difference is **used to subsidize the increase in Agent 2's consumption**.

# Standard Definitions

- Set of agents: $N = \{1, ..., n\}$
- A *consumption profile* (or *profile*) $\mathbf{q} = (q_1, ..., q_n) \in \mathbb{R}^n_+$
- Total consumption: $Q = \sum_{k=1}^{n} q_k$
- Cost function, $C : \mathbb{R}_+ \to \mathbb{R}_+$, $C(0) = 0$, **convex**
- A *cost-sharing rule*, $x : \mathbb{R}^n_+ \to \mathbb{R}^n$, assigns cost shares to achieve *budget balance:*

$$\sum_{i=1}^{n} x_i(\mathbf{q}) = C(Q).$$

# Fair Reward to the Frugal

**Axiom (Fair Reward to the Frugal, FRF)**

*For any* $\mathbf{q} \in \mathbb{R}^n_+$, *any* $i, j \in N$ *such that* $j \neq i$ *and* $q_i \leq Q/n \leq q_j$, *and any* $\mathbf{q}' \in \mathbb{R}^n_+$ *such that, for some* $\delta > 0$,

$$\begin{cases} q'_i = q_i - \delta \\ q'_j = q_j + \delta \\ q'_k = q_k & \text{for all } k \neq i, j \end{cases}$$

*we have:*

$$x_i(\mathbf{q}) - x_i(\mathbf{q}') \geq C(Q) - C(Q - \delta). \quad (1)$$

**GOAL:** To study the implications of **FRF**.

# FRF and Solidarity

# Solidarity

**Axiom (Solidarity, SOL)**

*For any* $\mathbf{q}, \mathbf{q}' \in \mathbb{R}^n_+$ *such that* $\mathbf{q}' \leq \mathbf{q}$, *then:*

$$x_i(\mathbf{q}) - x_i(\mathbf{q}') \geq 0 \qquad \forall i \in N. \tag{2}$$

Clear tension between **SOL** and **FRF**:

**Proposition**

*Let* $x$ *satisfy* **SOL** *and* $\mathbf{q}, \mathbf{q}' \in \mathbb{R}^n_+$ *such that* $\delta > 0$ *is swapped from agent* $i$*'s to agent* $j$*'s demand (*$q'_i = q_i - \delta$, $q'_j = q_j + \delta$ *and* $q'_k = q_k$ *for* $k \neq i, j$*), then:*

$$x_i(\mathbf{q}) - x_i(\mathbf{q}') \leq C(Q) - C(Q - \delta). \tag{3}$$

# Incompatibility between FRF and SOL

**Proposition**

***SOL** and **FRF** are incompatible unless $C$ is linear.*

**Note:** The linear case is a trivial one, where Solidarity actually has no bite/substance because the problem is separable.

# Proof (sketch)

Let $\mathbf{q} \in \mathbb{R}^n_+$ , $\delta > 0$, and define $q'_i = q_i - \delta$, $q'_j = q_j + \delta$ and $q'_k = q_k$ for $k \neq i, j$. By **FRF** and Prop 1,

$$x_i(\mathbf{q}) - x_i(\mathbf{q}') = C(Q) - C(Q - \delta). \tag{4}$$

Consider now going from $\mathbf{q}$ to $\mathbf{q}'$ in two steps. Define $\hat{q}_i = q_i - \delta/2$, $\hat{q}_j = q_j + \delta/2$ and $\hat{q}_k = q_k$ for $k \neq i, j$. By **FRF** and Prop 1,

$$\begin{cases} x_i(\mathbf{q}) - x_i(\hat{\mathbf{q}}) & = C(Q) - C(Q - \delta/2) \\ x_i(\hat{\mathbf{q}}) - x_i(\mathbf{q}') & = C(Q) - C(Q - \delta/2) \end{cases} \tag{5}$$

Summing up:

$$x_i(\mathbf{q}) - x_i(\mathbf{q}') = 2[C(Q) - C(Q - \delta/2)] \tag{6}$$

Comparing (4) and (6) yields a functional equation in $C$ whose solution must be linear.

# Why choose FRF over SOL?

**Context-dependent fairness**, to determine...

- which view of fairness to adopt**:** **FRF** or **SOL** ?
  - with convex costs, violations of **FRF** seem more unfair than violations of **SOL**

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  - **SOL** applies only when $Q$ changes
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**Our context:** We interpret $Q/n$ as the average wealth and/or needs of the population considered
$\implies$ Axioms relating to *other* populations (with $Q'/n = Q/n$) are unhelpful.

We will end up giving recommendations for all populations, but they will not rely on comparisons between dissimilar populations.

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**Why?** Because characterization proofs inevitably rely on these "impossible situations"
$\Longrightarrow$ This severely constrains the rules **for the situations we actually care about.**

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**Why?** Because characterization proofs inevitably rely on these "impossible situations"
$\Longrightarrow$ This severely constrains the rules **for the situations we actually care about.**
**Examples:** Dummy (cost/surplus allocation), Faithfulness (voting)

For practitioners to embrace our recommendations, we should avoid relying on such axioms to justify our solutions.

# Rules satisfying FRF

# A Lemma

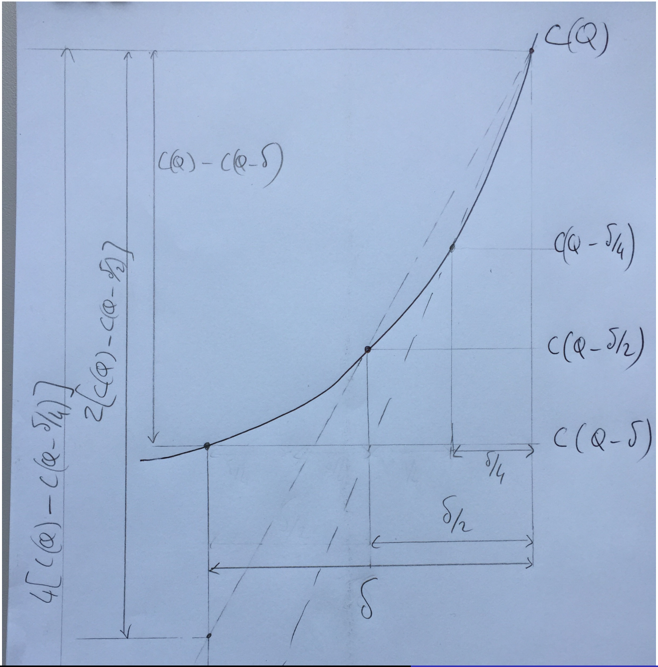
## Lemma

*A cost-sharing rule satisfies* **FRF** *if and only if for any* $\mathbf{q} \in \mathbb{R}^n_+$*, any* $i, j \in N$ *such that* $j \neq i$ *and* $q_i \leq Q/n \leq q_j$*, and any* $\mathbf{q}' \in \mathbb{R}^n_+$ *such that*

$$\begin{cases} q'_i = q_i - \delta \\ q'_j = q_j + \delta \\ q'_k = q_k & \text{for all } k \neq i, j \end{cases}$$

*for some* $\delta > 0$*:*

$$x_i(\mathbf{q}) - x_i(\mathbf{q}') \geq C'(Q)\,\delta. \tag{7}$$


C(Q)
C(Q) − C(Q − δ)
C(Q − δ/4)
C(Q − δ/2)
C(Q − δ)
δ/4
δ/2
δ
2[C(Q) − C(Q − δ/2)]
4[C(Q) − C(Q − δ/4)]

# Representation Theorem ($n = 2$)

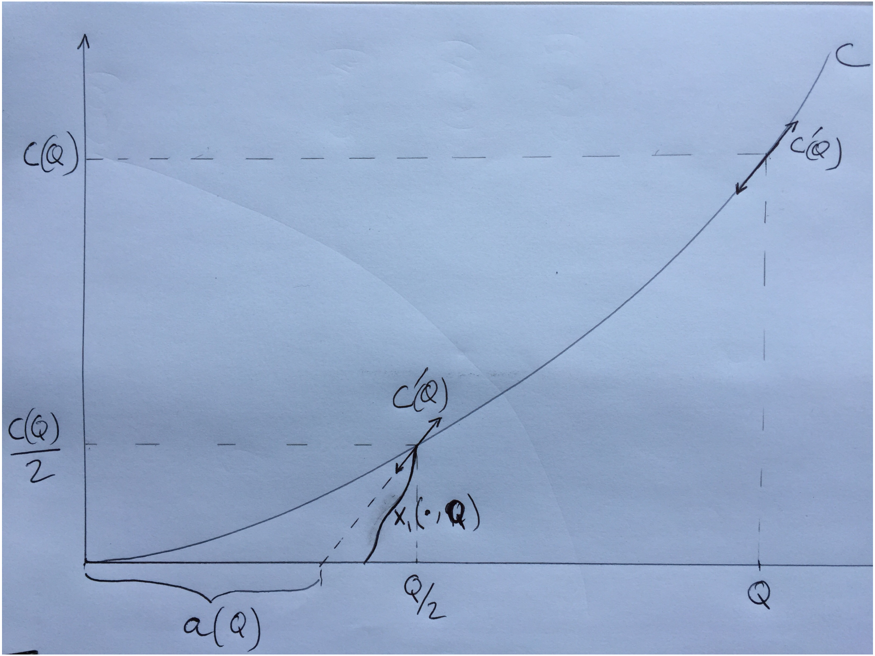
**Theorem**

*Let $n = 2$ and let $(q_1, q_2) \in \mathbb{R}^2_+$. Without loss of generality, label the agents such that $q_1 \leq q_2$. A cost-sharing rule $x$ satisfies* **FRF** *if and only if*

$$\begin{cases} x_1(q_1, q_2) & = x_1\left(\frac{Q}{2}, \frac{Q}{2}\right) - C'(Q) \int_{q_1}^{Q/2} (1 + \alpha(t, Q))\, dt \\ x_2(q_1, q_2) & = x_2\left(\frac{Q}{2}, \frac{Q}{2}\right) + C'(Q) \int_{q_1}^{Q/2} (1 + \alpha(t, Q))\, dt \end{cases} \tag{8}$$

*for some $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$.*

Conjecture for $n = 3$, using a generalization of **FRF** and "Outward Solidarity".


C
C(Q)
C'(Q)
C'(Q)
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2
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Q/2
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a(Q)

# *With many agents ($n \geq 3$)... and using heavy artillery

**Axiom (Anonymity, ANO)**

*For all permutations $\sigma : N \to N$, and all demand profiles $\mathbf{q} \in \mathbb{R}^n_+$,*

$$x_i(\mathbf{q}) = x_{\sigma(i)}(\mathbf{q}^\sigma) \qquad \text{for all } i = 1, ..., n,$$

*where $\mathbf{q}^\sigma \equiv \left(q_{\sigma(i)}\right)_{i \in N}$.*

**Axiom (No Advantageous Reallocation, NAR)**

*For any coalition $T \subseteq N$ and any demand profiles $\mathbf{q}, \mathbf{q}' \in \mathbb{R}^n_+$ such that*

$$\sum_{j \in T} q'_j = \sum_{j \in T} q_j, \tag{9}$$

*and $q'_k = q_k$ for all $k \in N \backslash T$, then*

$$\sum_{j \in T} x_j(\mathbf{q}') = \sum_{j \in T} x_j(\mathbf{q}). \tag{10}$$

# *Representation Theorem $n \geq 3$

**Proposition**

*Let $|N| \geq 3$. A cost-sharing rule $x$ satisfies* **FRF**, **ANO**, *and* **NAR**, *if and only if*

$$x_i(\mathbf{q}) = \frac{C(Q)}{n} + \left(q_i - \frac{Q}{n}\right) P(Q),$$

*for all $i \in N$, with $P(Q) \geq C'(Q)$.*

**Intuition of proof:**

- **ANO** implies **ETE**: $x_i(Q/n, ..., Q/n) = C(Q)/n$
- **ANO** + **NAR** imply the linearity of $x$: departure from $Q/n$ at a common price $P(Q)$
- Lemma 1 implies $P(Q) \geq C'(Q)$.

# Minimizing Free Consumption

# Negative cost shares are costly

Although they give an incentive for some small users to further reduce their consumption, they imply that subsidies must be handed even to agents who derive little utility from consuming the good (and who would otherwise not have consumed much anyway).

**Axiom (Restricted Fair Reward to the Frugal, RFRF)**

*For any $\mathbf{q} \in \mathbb{R}_+^n$, any $i, j \in N$ such that $j \neq i$ and $q_i \leq Q/n \leq q_j$, and any $\mathbf{q}' \in \mathbb{R}_+^n$ such that*

$$\begin{cases} q_i' = q_i - \delta \\ q_j' = q_j + \delta \\ q_k' = q_k & \text{for all } k \neq i, j \end{cases}$$

*for some $\delta > 0$:*

$$0 \leq x_i(\mathbf{q}') \leq \max\{x_i(\mathbf{q}) - [C(Q) - C(Q - \delta)], 0\}. \tag{11}$$

**Theorem**

*Let $n = 2$ and let $(q_1, q_2) \in \mathbb{R}^2_+$. Without loss of generality, label the agents such that $q_1 \leq q_2$. A* **non-negative cost-sharing rule** *$x$ satisfies* **RFRF** *if and only if*

$$\begin{cases} x_1(q_1, q_2) & = \max\left\{x_1\left(\frac{Q}{2}, \frac{Q}{2}\right) - C'(Q)\int_{q_1}^{Q/2}(1 + \alpha(t, Q))\, dt, 0\right\} \\ x_2(q_1, q_2) & = C(Q) - x_1(q_1, q_2) \end{cases} \tag{12}$$

*for some $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, with $x_1(Q/2, Q/2) \in [0, C(Q)]$.*

If $x_1\left(\frac{Q}{2}, \frac{Q}{2}\right)$ is "not too large", **RFRF** implies that some units are given away or free.
(Always true with **ETE**, where $x_1\left(\frac{Q}{2}, \frac{Q}{2}\right) = C(Q)/2$.)

# Minimizing Free Lunches

## Theorem

*A non-negative cost-sharing rule $x$ that satisfies* **RFRF** *and* **ETE** **minimizes the number of units given away for free** *if and only if*

$$x_i(\mathbf{q}) = \max\left\{\frac{C(Q)}{n} + \left(q_i - \frac{Q}{n}\right) C'(Q), 0\right\}.$$

**Proof:** OK for $n = 2$, conjecture for $n \geq 3$ (need more time)

The quantity $a(Q) = \frac{Q}{n}\left(1 - \frac{C(Q)/Q}{C'(Q)}\right)$ is the (minimum) amount given away for free.

# Efficiency

(very much in progress)

**Axiom (Efficiency, EFF)**

*A cost-sharing rule x is efficient if*

$$\left.\frac{\partial x_i(\mathbf{q})}{\partial q_i}\right|_{Q=\bar{Q}} = C'\left(\bar{Q}\right) \qquad \forall i \in N,$$

*for some common* $\bar{Q} \in \mathbb{R}_+$.

# EFF implies a two-part tariff

**Conjecture**

*A non-negative cost-sharing rule $x$ that satisfies* **EFF** *and* **ETE** *if and only if*

$$x_i(\mathbf{q}) = \max\left\{\frac{C(Q)}{n} + \left(q_i - \frac{Q}{n}\right) C'(Q), 0\right\}.$$

**Intuition of proof:**

- **EFF** implies $x_i(\mathbf{q}) = \max\{\alpha_i(Q) + \beta_i(q_i, Q) C'(Q), 0\}$
- **ETE** implies $\alpha_i(Q) = \alpha_j(Q) = \alpha(Q)$ and $\beta_i(\cdot, Q) = \beta_j(\cdot, Q) = \beta(\cdot, Q)$ for all $i, j$.
- Budget balance $\alpha(Q) = C(Q)/n$ and $\beta(\cdot, Q) = q_i - Q/n$

# Implications

- Efficiency **requires** giving away units for free
- Fairness justification of Efficiency: Efficiency requires satisfying **RFRF** (and giving away the least amount of units for free)

# Conclusion

- **FRF** requires rewarding consumption reductions at a rate that is no less than the marginal cost
  - we lose: **SOL**
  - we gain: "more appropriate" fairness + **EFF**
- **SOL** is not as inoccuous an axiom as one might think
  - "context-dependent fairness"
- **(R)FRF** and **EFF** both point toward a two-part tariff:
  - split the total cost equally: $C(Q)/n$
  - charge/reward departures from average consumption at a rate equal to $C'(Q)$