Judgement Aggregation and Strategy-Proof Social Choice

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Outline

Literature on Judgement Aggregation

Precursors

G.T.Guilbaud: 'Theories of the General Interest, and the Logical Problem of Aggregation,' 1966.

R.Wilson, 'On the Theory of Aggregation,' 1975.

A.Rubinstein and P.Fishburn, 'Algebraic Aggregation Theory,' 1986.

Seminal work

L.A.Kornhauser and L.G.Sager, 'Unpacking the Court,' 1986.

C.List and P.Pettit, 'Aggregating Sets of Judgements: An Impossibility Result,' 2002.

K.Nehring and C.Puppe, 'Strategy-Proof Social Choice on Generalized Single-Peaked Domains,'2002.

G.Pigozzi, 'Belief merging and the discursive dilemma,' 2006.

Issue-wise aggregation

F.Dietrich and C.List, 'Arrow's Theorem in Judgement Aggregation,' 2007.

E.Dokow and R.Holzman, 'Aggregation of Binary Evaluations,' 2010.

K.Nehring and C.Puppe, 'The Structure of Strategy-Proof Social Choice. Part I: ... ,' 2007.

K.Nehring and C.Puppe, 'Abstract Arrovian Aggregation,'2010.

F.Dietrich and C.List, 'Propositionwise Judgement Aggregation: The General Case,' 2013.

Literature on Judgement Aggregation (continued)

Premise-Based versus Conclusion-Based Procedures

K.Nehring, 'The Impossibility of a Paretian Rational,' 2005.

P.Mongin, 'Factoring out the Impossibility of Logical Aggregation,' 2008.

F.Dietrich and P.Mongin, 'The Premise-Based Approach to Judgement Aggregation,' 2010.

General Aggregation without Independence

C.List, 'A Model of Path-Dependence in Decisions over Multiple Propositions,' 2004.

K.Nehring, M.Pivato and C.Puppe, 'The Condorcet Set: Maj. Voting over Intercon. Prop.', 2014.

F.Dietrich, 'Scoring Rules for Judgement Aggregation,' 2014.

Surveys

C.List and C.Puppe, 'Judgement Aggregation: A Survey,' 2009.

C.List, 'The Theory of Judgement Aggregation. An Introductory Review ,' 2012.

... and many more, especially since 2010 in Economics and Computer Science.

Basic Definitions and Notation

- Consider alternatives that can be described in **binary code**:
- There are K binary issues ('propositions') which can take on the value 1 (yes') or 0 ('no').
- Thus, an **alternative** is a binary sequence of length K:

$$x = (x^1, ..., x^K).$$

- There may be restrictions, i.e. not *all* binary sequences may be feasible. Let

$$X \subseteq \{0, 1\}^K$$

  denote the set of **feasible alternatives**.
- Sometimes, we will refer to a (feasible) alternative as a (feasible) **view**.

Example: Asymmetric Binary Relations

- Suppose that there is a set of three candidates $A = \{a, b, c\}$, with the following three issues:
  - Issue 1: 'a better than b'
  - Issue 2: 'b better than c'
  - Issue 3: 'c better than a'
- Assume that (binary) preference judgements are connected and asymmetric, i.e. negating the statement 'a better than b' means 'b better than a.'
- Then, an alternative (a 'view') is a connected and asymmetric binary relation, i.e. a complete and directed graph.
- For instance, the view $(1, 1, 0)$ corresponds to the binary relation $\{(a, b), (b, c), (a, c)\} \subseteq A^2$, i.e. to the *transitive* preference ordering $a \succ b \succ c$.
- By contrast, the view $(1, 1, 1)$ corresponds to the binary relation $\{(a, b), (b, c), (c, a)\} \subseteq A^2$, i.e. to the *cyclic* relation $a \succ b$, $b \succ c$, $c \succ a$.

Example: Strict Preferences

- With the set of candidates $A = \{a, b, c\}$ and the issues:
  - Issue 1: 'a better than b'
  - Issue 2: 'b better than c'
  - Issue 3: 'c better than a'

  the space of all **linear preference orderings** (i.e. asymmetric, transitive and connected binary relations) is given by the feasible set

$$X_A^{lin} = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}.$$

- In general, let X_A^{lin} denote the space of all linear preference orderings over the finite set A of candidates.

Example: General Reflexive Binary Relations

- Again, let $A = \{a, b, c\}$, but now with the following six issues:
  - Issue 1: 'a at least as good as b'
  - Issue 2: 'b at least as good as a'
  - Issue 3: 'b at least as good as c'
  - Issue 4: 'c at least as good as b'
  - Issue 5: 'a at least as good as c'
  - Issue 6: 'c at least as good as a'
- A view is now a general, possibly incomplete, binary relation.
- By convention, always assume reflexivity.
- For instance, the view $(1, 1, 0, 0, 0, 0)$ corresponds to the (weak) **partial order** that declares a and b as indifferent, and all other pairs of alternatives as incomparable.
- The view $(1, 1, 1, 0, 1, 0)$ corresponds to the **weak order** $a \sim b \succ c$.

The Doctrinal Paradox

	p	q	$d = p \wedge q$
Judge 1	yes	no	no
Judge 2	no	yes	no
Judge 3	yes	yes	yes
Majority	yes	yes	yes or no?

- p : defendant had a contractual obligation
- q : the contract was legally valid
- $d = p \wedge q$: legal doctrine

Kornhauser and Sager, 1986; Pettit, 2001; List and Pettit, 2002, 2011.

Example: Truth-Functional Decisions

- A binary decision d is truth-functionally determined by a set of 'premises' $\{p_1, ..., p_m\}$, i.e. there are $m + 1$ issues.
- For instance, $d = p \wedge q$ as in the doctrinal paradox; if p corresponds to issue 1, q to issue 2 and $d = p \wedge q$ to issue 3, the set of feasible views is given by

$$X^d_{p \wedge q} = \{(0,0,0),(0,1,0),(1,0,0),(1,1,1)\}$$

- As another example, consider $d = p \leftrightarrow q$; then,

$$X^d_{p \leftrightarrow q} = \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$$

Dokow and Holzman, 2009; Mongin, 2008; Nehring and Puppe, 2008.

Example: Selecting Members of a Committee

- K candidates for membership in a committee. For each candidate, the question is: should $k \in K$ be selected as a member, or not?
- Suppose that the committee has to contain at least I and at most J members, where $0 \leq I \leq J \leq K$, then

$$X_{K;I,J} = \{x \in \{0,1\}^K \ : I \leq ||x|| \leq J\},$$

  where $||x|| = \sum_{k=1}^{K} x^k$.
- For instance,

$$\begin{aligned} X_{3;1,2} &= \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\}, \\ X_{3;1,3} &= \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}. \end{aligned}$$

Example: Resource Allocation

- Suppose that there is a money amount $M \geq 0$ that has to be spent on L public goods. If $\phi^\ell \geq 0$ is the amount spent on public good ℓ, feasibility requires

$$\sum_{\ell=1}^{L} \phi^\ell = M$$

- Binary issues: 'spend at least j cents on good ℓ?' for $j = 1, ..., M$ and $\ell = 1, ..., L$.
- For instance, with $M = 4ct$ and $L = 3$ goods, the allocation that assigns $2ct$ to good 1, and $1ct$ to goods 2 and 3, respectively, corresponds to the binary sequence

$$(1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0) \ .$$

Critical Fragments

Unless otherwise noted, all following concepts and results are taken from Nehring and Puppe, 2002, 2007 and 2010.

- A 'partial' view $w \in \{0,1\}^J$ where $J \subseteq \{1, ..., K\}$ is called a **fragment**; the set J is called the *support* of w, and $\#J$ is the *length* of w.
- A fragment w is **forbidden** (given X) if there is no feasible view $x \in X$ such that x coincides with w on its support.
- A fragment w is called **critical** (given X) if it is minimally forbidden, i.e. if it is forbidden, and no proper subfragment of w is forbidden.
- We will assume that X is in no issue *trivial*, i.e. for all $k = 1, ..., K$ there exist $x, y \in X$ such that $x^k = 0$ and $y^k = 1$. In other words, fragments of length 1 are never forbidden.

Aggregation

- Let $N = \{1, ..., n\}$ be a (finite) set of individuals.
- An **aggregation rule** is a mapping $f : X^n \to \{0,1\}^K$. The view $f(x_1, ..., x_n)$ is the *collective view* corresponding to the profile $(x_1, ..., x_n)$ of individual views.
- For instance, with n odd, the **(issue-wise) majority rule** f_{maj} is defined as follows. For all $k = 1, ..., K$,

$$[f_{maj}(x_1, ..., x_n)]^k = \begin{cases} 0 & \text{if} \quad \#\{i \in N \ : \ x_i^k = 0\} > \frac{n}{2} \\ 1 & \text{if} \quad \#\{i \in N \ : \ x_i^k = 0\} < \frac{n}{2} \end{cases}$$

- An aggregation rule f is **consistent** if $f(X^n) \subseteq X$, i.e. if f produces a feasible collective view for all profiles of feasible individual views.

Consistency of Issue-Wise Majority Voting

Theorem

Let $X \subseteq \{0,1\}^K$ be a space of feasible views. Then, the issue-wise majority rule f_{maj} with an odd number of individuals is consistent on X if and only if all critical fragments of X have length 2.

Notation: If w is a critical fragment, denote by w^{-j} the (non-forbidden) fragment that results from w by negating the j-th issue. Moreover, write $x \sqsupset w^{-j}$ if $x \in X$ extends w^{-j}.

Proof *(of necessity). Suppose that w is a critical fragment of length > 2. W.l.o.g. suppose that $w = (w^1, w^2, w^3, *, ..., *)$. By criticality, there exist $x, x', x'' \in X$ such that $x \sqsupset w^{-1}$, $x' \sqsupset w^{-2}$ and $x'' \sqsupset w^{-3}$. If $1/3$ of the population endorses x, x', x'', respectively, f_{maj} yields w on its support. Thus, f_{maj} is not consistent on X.*

Median Spaces

- Given a space $X \subseteq \{0,1\}^K$ of feasible views and three elements $x, y, z \in X$, say that y is (weakly) **between** x and z, denoted by $y \in [x, z]$, if y coincides with x and z in all issues in which they coincide, i.e. if, for all $k = 1, ..., K$,

$$x^k = z^k \;\Rightarrow\; y^k = x^k = z^k.$$

- Geometrically, y is between x and z if and only if y is contained in the 'subcube' spanned by x and z.
- A space $X \subseteq \{0,1\}^K$ is called a **median space** if any triple of elements $x, y, z \in X$ admits an element $x_{med} \in X$, their *median*, that is between any pair of the triple, i.e.

$$x_{med} \in [x, y] \cap [x, z] \cap [y, z].$$

Median Spaces: Characterization

Observation

If a triple admits a median, then the median is uniquely determined.

Proposition

A space $X \subseteq \{0,1\}^K$ of feasible views is a median space if and only if all critical fragments of X have length 2.

Proof. *Suppose $w = (w^1, w^2, w^3, *, \ldots, *)$ is a critical fragment of length > 2. Let $x, y, z, \in X$ be such that $x \sqsupset w^{-1}$, $y \sqsupset w^{-2}$ and $z \sqsupset w^{-3}$, then $[x,y] \cap [x,z] \cap [y,z] \cap X = \emptyset$, i.e. X is not a median space.*

Conversely, suppose $x, y, z \in X$ are such that $[x,y] \cap [x,z] \cap [y,z] \cap X = \emptyset$. W.l.o.g. we may assume that $x = (1,0,0,*,\ldots,*)$, $y = (0,1,0,*,\ldots,*)$ and $z = (0,0,1,*,\ldots,*)$. But then there is a critical fragment containing $(0,0,0)$.

Properties of Aggregation Rules

In the following, let $f : X^n \to \{0,1\}^K$ be an aggregation rule.

- f is **consistent** if $f(X^n) \subseteq X$
- f satisfies **sovereignty** if $f(X^n) \supseteq X$.
- f satisfies **unanimity** if, for all $x \in X$, $f(x, ..., x) = x$.
- f satisfies **independence** if, for all $(x_1, ..., x_n), (y_1, ..., y_n) \in X^n$, and all $k = 1, ..., K$,

$$\left(\text{for all } i \in N,\ x_i^k = y_i^k\right) \Rightarrow [f(x_1, ..., x_n)]^k = [f(y_1, ..., y_n)]^k.$$

Properties of Aggregation Rules
(continued)

- f satisfies **positive responsiveness** if, for all $x_1, ..., x_n$, all $i \in N$, all $k = 1, ..., K$, and all $\alpha \in \{0, 1\}$,

$$[f(x_1, ..., x_i, ..., x_n)]^k = \alpha \;\Rightarrow\; [f(x_1, ..., x(\alpha)_i, ..., x_n)]^k = \alpha,$$

  where $x(\alpha)_i^\ell = x_i^\ell$ for all $\ell \neq k$, $x(\alpha)_i^k = \alpha$ if this is compatible with feasibility, and $x(\alpha)_i^k = x_i^k$ otherwise.

- f satisfies **monotone independence** if, for all $(x_1, ..., x_n), (y_1, ..., y_n) \in X^n$, all $k = 1, ..., K$, and all $\alpha \in \{0, 1\}$,

$$\left(\text{for all } i \in N,\; x_i^k = \alpha \Rightarrow y_i^k = \alpha\right)$$

$$\text{implies} \quad [f(x_1, ..., x_n)]^k = \alpha \Rightarrow [f(y_1, ..., y_n)]^k = \alpha.$$

Winning Coalitions of Agents

Definition (Families of Winning Coalitions)

A family of **winning coalitions** of agents is a non-empty collection of non-empty subsets of agents that is closed under taking supersets (i.e. if $W \subseteq N$ is winning, then W' is winning for all $W' \supseteq W$).

Definition (Structure of Winning Coalitions)

A **structure of winning coalitions** $\mathcal{W}$ assigns two families of winning coalitions $\mathcal{W}^k_\alpha, \mathcal{W}^k_{1-\alpha}$, $\alpha \in \{0, 1\}$, to each issue k such that, for all $k \in K$:

$$W \in \mathcal{W}^k_\alpha \Leftrightarrow (N \backslash W) \notin \mathcal{W}^k_{1-\alpha}. \qquad (*)$$

Intuition:

- The coalitions W in $\mathcal{W}^{\boldsymbol{k}}_{\boldsymbol{0}}$ are 'winning' for 0 in issue k in the sense that if all members of W agree on 0 in issue k they can force this as the collective result.
- Similarly, the coalitions W in $\mathcal{W}^{\boldsymbol{k}}_{\boldsymbol{1}}$ are 'winning' for 1 in issue k.
- "$\Rightarrow$" of condition (*) says that the set of agents N cannot be partitioned into a coalition that is winning for 1 and a coalition that is winning for 0 in the same issue.
- "$\Leftarrow$" of condition (*) says that at least one coalition in a bi-partition of N must be winning either for 1 or for 0 in each issue.

Issue-Wise Aggregation: 'Voting by Issues'

Definition (Voting by Issues)

Voting by issues (with the structure $\mathcal{W}$ of winning coalitions) is the aggregation rule $f : X^n \to \{0,1\}^K$ defined as follows. For all $(x_1, \dots, x_n)$, $k \in K$ and $\alpha \in \{0,1\}$,

$$[f(x_1, \dots, x_n)]^k = \alpha \;\Leftrightarrow\; \{i \in N : x_i^k = \alpha\} \in \mathcal{W}_\alpha^k.$$

Theorem

An aggregation rule $f : X^n \to X$ satisfies sovereignty and monotone independence if and only if it is (consistent) voting by issues.

(Proof: Exercise)

The Intersection Property

Definition (Intersection Property)

A structure of winning coalitions $\mathcal{W}$ satisfies the **Intersection Property** if, for every critical fragment $w = (w^1, \ldots, w^J)$ on $J \subseteq K$, and every selection $W^j \in \mathcal{W}^j_{w^j}$ for all $j \in J$,

$$\bigcap_{j \in J} W^j \neq \emptyset$$

Theorem

Voting by issues with structure of winning coalitions $\mathcal{W}$ is consistent if and only if $\mathcal{W}$ satisfies the Intersection Property.

Intersection Property: Proof

Proof. "$\Rightarrow$" *(by contraposition) Let* $w = (w^1, \ldots, w^J)$ *be a critical fragment. For all* $j \in J$ *consider any selection* $W^j \in \mathcal{W}^j_{w^j}$. *Suppose that* $\bigcap_{j \in J} W^j = \emptyset$. *Then, for all* $i \in N$, *there exists* j_i *such that* $i \notin W^{j_i}$. *For each* i *pick a feasible* $x_i \in X$ *such that* $x_i \sqsupset w^{-j_i}$. *By construction, if* $i \in W^j$ *then* $j \neq j_i$, *hence* $x_i^j = w^j$, *i.e.* $W^j \subseteq \{i : x_i^j = w^j\}$. *Thus,* $\{i : x_i^j = w^j\} \in \mathcal{W}^j_{w_j}$ *for all* $j \in J$, *i.e. voting by issues with the given structure of winning coalitions is inconsistent.*

"$\Leftarrow$" *(by contraposition) Let voting by issues with* $\mathcal{W}$ *be inconsistent, i.e. for some* $(x_1, \ldots, x_n)$, *we have* $f(x_1, \ldots, x_n) \notin X$. *Then, there exists a critical fragment* $w = (w_1, \ldots, w_J)$ *such that* $w \sqsubseteq f(x_1, \ldots, x_n) \notin X$. *Suppose now that the Intersection Property is satisfied, and let* $W^j \in \mathcal{W}^j_{w^j}$ *be a selection of winning coalitions for all* $j \in J$. *Since* $\bigcap_{j \in J} W^j \neq \emptyset$, *there exists* $i_0 \in W^j$ *for all* $j \in J$, *but then* $x_{i_0} \sqsupseteq w$, *contradicting the fact that* w *is a critical fragment.*

Conditional Entailment

Definition

For all $\alpha, \alpha' \in \{0,1\}$ and distinct $k, k' \in K$, say that (k,α) **directly conditionally entails** (k',α'), written as $(k,\alpha) \geq^0 (k',\alpha')$, if there exists a critical fragment w such that $w^k = \alpha$ and $w^{k'} = 1 - \alpha'$. Moreover, denote by $\geq$ the transitive closure of $\geq^0$, and say that (k,α) **conditionally entails** (k',α') if $(k,\alpha) \geq (k',\alpha')$.

Intuition:

- $(k,\alpha) \geq^{\mathbf{0}} (k',\alpha')$ means that, fixing some other issues in the way prescribed by some critical fragment w, α in issue k is inconsistent with $1-\alpha'$ in issue k'.

Observation

As already noted, in median spaces all critical fragments have length 2. This means that all entailments are <u>*unconditional*</u>*. Is it also true that that all entailments are direct? (Proof or counterexample: Exercise).*

Contagion

Lemma

Suppose that the structure of winning coalitions $\mathcal{W}$ satisfies the Intersection Property, and $(k, \alpha) \geq (k', \alpha')$, then $\mathcal{W}^k_\alpha \subseteq \mathcal{W}^{k'}_{\alpha'}$.

Proof. *By transitivity, it suffices to show that $(k, \alpha) \geq^0 (k', \alpha')$ implies $\mathcal{W}^k_\alpha \subseteq \mathcal{W}^{k'}_{\alpha'}$. Thus, let $w = (w^1, ..., w^J)$ be a critical fragment with $w^k = \alpha$ and $w^{k'} = 1 - \alpha'$, and consider any $W \in \mathcal{W}^k_\alpha$ and any $W' \in \mathcal{W}^{k'}_{1-\alpha'}$. By the Intersection Property, $W \cap W' \neq \emptyset$. Thus, by the following observation, $\mathcal{W}^k_\alpha \subseteq \mathcal{W}^{k'}_{\alpha'}$.*

Observation

By condition (*) above, we have, for all $k \in K$,

$$\mathcal{W}^k_\alpha = \{W \subseteq N \;:\; W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}^k_{1-\alpha}\}. \qquad (**)$$

Veto Lemma

Lemma

Suppose that the structure of winning coalitions $\mathcal{W}$ satisfies the Intersection Property, and that there exists a critical fragment of length ≥ 3, say $w = (w^1, w^2, w^3, *, \ldots, *)$. If $\mathcal{W}^1_{1-w^1} \subseteq \mathcal{W}^2_{w^2}$, then $\{i\} \in \mathcal{W}^3_{1-w^3}$ for some agent $i \in N$.

Proof. *Let $\tilde{W}_1$ be a minimal element of $\mathcal{W}^1_{w^1}$, and let $i \in \tilde{W}_1$. By (**), $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}^1_{1-w^1}$. By assumption, $\mathcal{W}^1_{1-w^1} \subseteq \mathcal{W}^2_{w^2}$, hence $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}^1_{w^2}$. Consider any $W_3 \in \mathcal{W}^3_{w^3}$; by the Intersection Property, $\tilde{W}_1 \cap (\tilde{W}_1^c \cup \{i\}) \cap W_3 \neq \emptyset$. But this implies that $i \in W_3$ for all $W_3 \in \mathcal{W}^3_{w^3}$, hence by (**), $\{i\} \in \mathcal{W}^3_{1-w^3}$.*

Main Impossibility Result

Definition (Total Blockedness)

An aggregation space X is called **totally blocked** if the conditional entailment relation is complete, i.e. if for all issues k, k' and all $\alpha, \alpha' \in \{0, 1\}$, $(k, \alpha) \geq (k', \alpha')$.

Theorem

An aggregation space X admits non-dictatorial aggregation rules that are monotonely independent, sovereign and consistent if and only if X is not totally blocked.

Proof of Main Impossibility Theorem

Proof. *It is easily seen that total blockedness of* X *implies the existence of a critical fragment of length* ≥ 3*. By the contagion lemma,* $\mathcal{W}_{\alpha}^{k} = \mathcal{W}_{\alpha'}^{k'}$ *for all* $k, k' \in K$ *and all* $\alpha, \alpha' \in \{0,1\}$*. By the veto lemma, there exists* $i \in N$ *who has a veto, hence in fact is a dictator.*

Conversely, suppose that X *is not totally blocked. Define*

$$\begin{aligned} K^0 &:= \{k \in K \ : \ (k,1) \equiv (k,0)\}, \\ K^+ &:= \{k \in K \ : \ (k,1) > (k,0)\}, \\ K^- &:= \{k \in K \ : \ (k,0) > (k,1)\}, \\ K^* &:= \{k \in K \ : \ \textit{neither } (k,1) \geq (k,0) \textit{ nor } (k,0) \geq (k,1)\}. \end{aligned}$$

Clearly, $\{K^0, K^+, K^-, K^*\}$ *forms a partition on* K*.*

Case 1: *If* $K^+ \cup K^- \neq \emptyset$*, then set* $\mathcal{W}_0^k = 2^N \setminus \{\emptyset\}$ *and* $\mathcal{W}_1^k = \{N\}$ *if* $k \in K^+$*, and* $\mathcal{W}_0^k = \{N\}$ *and* $\mathcal{W}_1^k = 2^N \setminus \{\emptyset\}$ *if* $k \in K^-$*. Moroever, choose a voter* $i \in N$ *and set* $\mathcal{W}_0^k = \mathcal{W}_1^k = \{W \subseteq N \ : \ i \in W\}$ *for all* $k \in K^0 \cup K^*$ *(if the latter set is non-empty). Clearly, this defines a non-dictatorial rule. It can be verified that the Intersection Property is satisfied.*

Proof of Main Impossibility Theorem
(continued)

Case 2: *Suppose* $K^+ \cup K^- = \emptyset$ *and that both* K^0 *and* K^* *are non-empty. Then, specify two different "local" dictators* i *and* j *on* K^0 *and* K^**, respectively. One can show that every critical fragment must have support either entirely in* K^0*, or entirely in* K^**. Hence, by the Intersection Property, the rule just defined is consistent and non-dictatorial.*

Case 3: *Suppose now that* K^* *is also empty, i.e.* $K = K^0$*. Since* X *is not totally blocked,* K *is partitioned into at least two equivalence classes with respect to the equivalence relation* $\equiv$*. Since, obviously no critical fragment can meet two different equivalence classes, we can specify different dictators on different equivalence classes while satisfying the Intersection Property.*

Case 4: *Suppose finally that* K^0 *is also empty, i.e.* $K = K^*$*. Then one can show that there exists a view* $x \in X$ *such that any critical fragment coincides with* x *in at most one issue. Using the Intersection Property, this implies that the (non-dictatorial) unanimity rule defined by* $\mathcal{W}^k_{x^k} = 2^N \setminus \{\emptyset\}$ *and* $\mathcal{W}^k_{1-x^k} = \{N\}$*, for all* $k \in K$*, is consistent.*

Examples of Dictatorial Domains

Proposition (with Arrow's Theorem as Corollary)

The space X_A^{lin} is totally blocked (Nehring, 2003). The space X_A^{weak} is totally blocked (Dietrich and List, 2007).

Proposition

For all $K \geq 3$, the spaces $X_{K;1,K-1}$ are totally blocked.

Proposition

The resource allocation problem is totally blocked if $L \geq 3$.

Oligarchies

Definition (Semi-Blockedness)

An aggregation space X is called **semi-blocked**, if for all issues k, k' and all $\alpha, \alpha' \in \{0, 1\}$, either $(k, \alpha) = (k', \alpha')$ or $(k, \alpha) = (k', 1 - \alpha')$, where '$=$' is the symmetric part of the conditional entailment relation.

Theorem (Nehring, 2006)

An aggregation space X admits non-oligarchic aggregation rules that are monotonely independent, sovereign and consistent if and only if X is not semi-blocked.

Examples of semi-blocked aggregation spaces: partial orders, equivalence relations, all truth-functional decisions.

Existence of Anonymous Aggregation Rules

Definition (Blockedness)

An aggregation space X is called **blocked**, if for some issue k, $(k, \alpha) = (k, 1 - \alpha)$.

Theorem

An aggregation space X admits, for all n, anonymous aggregation rules that are monotonely independent, sovereign and consistent if and only if X is not blocked.

Examples of non-blocked aggregation spaces: all median spaces, the spaces $X_{K;1,K}$ and $X_{K;0,K-1}$.

Generalized Single-Peaked Preferences

- Now interpret feasible views as alternatives, and assume that individuals have **preferences** over the set X of feasible views.
- Since X admits a notion of 'betweenness,' we can define:

Definition

A linear ordering $\succ_i$ with top element $x_i^* \in X$ is **(generalized) single-peaked on** X if, for all distinct views $y, z \in X$,

$$y \in [x_i^*, z] \;\Rightarrow\; y \succ_i z.$$

Let $\mathcal{S}(X)$ denote set of all generalized single-peaked orderings on X.

Remark

All what follows remains valid if one replaces the space $\mathcal{S}(X)$ of **all** *generalized single-peaked preferences by a* **rich** *domain of generalized single-peaked pref. In fact, all results hold with* **weak** *orders that admit a* **unique** *top alternative.*

Special Cases

- If X 'linear,' then $\mathcal{S}(X)$ standard space of **single-peaked** preferences (Black, 1948; Arrow, 1951; Moulin, 1980).
- If $X = \{0,1\}^K$, then $\mathcal{S}(X)$ space of **separable** preferences (Barberà, Sonnenschein, Zhou, 1991).
- If $X = \{e_k\}_{k \in K}$, where e_k is the k-th unit vector $(0,..,0,1,0,...,0)$, then $\mathcal{S}(X)$ is the **unrestricted domain**.
- If X 'cyclic', then $\mathcal{S}(X)$ space of preferences that are **single-peaked on a circle** (Schummer and Vohra, 2002).
- If X 'multi-dimensionally linear,' then $\mathcal{S}(X)$ space of **multi-dimensionally single-peaked** preferences (Barberà, Gul, Stacchetti, 1993).

Strategy-Proof Social Choice Functions

Definition

A **social choice function (scf)** is a mapping $F : \mathcal{D}^n \to X$ that assigns an alternative to each profile of individual preferences from some domain $\mathcal{D}$.

A scf F is **strategy-proof on** $\mathcal{D}$ if, for all $i \in N$, all $(\succ_1, ..., \succ_n) \in \mathcal{D}^n$, and all $\succ'_i \in \mathcal{D}$,

$$F(\succ_1, ..., \succ_i, ..., \succ_n) \ \succcurlyeq_i \ F(\succ_1, ..., \succ'_i, ..., \succ_n).$$

A scf F is **sovereign** if $F(\mathcal{D}^n) = X$, and it is **dictatorial** if there exists $h \in N$ such that, for all $(\succ_1, ..., \succ_n)$,

$$F(\succ_1, ..., \succ_n) = \ x^*_h \ := \ \text{top alternative of } \succ_h \ .$$

'Tops Only'

Proposition ('Tops-onlyness')

Suppose that $F : \mathcal{S}(X)^n \to X$ is sovereign and strategy-proof, then F depends only on the vector $(x_1^*, ..., x_n^*)$ of the respective top alternatives of $(\succ_1, ..., \succ_n)$.

based on: Barberà, Massó and Neme, 1997.

Corollary (Representation by an aggregation function)

Thus, every sovereign and strategy-proof scf $F : \mathcal{S}(X)^n \to X$ can be represented by an aggregation function $f : X^n \to X$ such that

$$F(\succ_1, ..., \succ_n) \; = \; f(x_1^*, ..., x_n^*),$$

where $(x_1^*, ..., x_n^*)$ are the top alternatives of $(\succ_1, ..., \succ_n)$.

Strategy-Proofness is Equivalent to Monotone Independence

Theorem

Let $F : \mathcal{S}(X)^n \to X$ be represented by the aggregation function $f : X^n \to X$. Then, F is sovereign and strategy-proof if and only if f is sovereign and monotonely independent.

The Gibbard-Satterthwaite Theorem Generalized

Theorem

The generalized single-peaked domain $\mathcal{S}(X)$ admits non-dictatorial, sovereign and strategy-proof social choice functions if and only if X is not totally blocked.

Corollary (The Gibbard-Satterthwaite Theorem)

If X has at least three elements, every sovereign and strategy-proof scf over the unrestricted preference domain on X is dictatorial.

Proof. *The unrestricted domain is generalized single-peaked on the space $X = \{e_k\}_{k \in K}$,which is totally blocked if $K \geq 3$.*

More Domain Characterization Results

Theorem

The generalized single-peaked domain $\mathcal{S}(X)$ admits non-oligarchic, sovereign and strategy-proof social choice functions if and only if X is not semi-blocked.

Theorem

The generalized single-peaked domain $\mathcal{S}(X)$ admits, for each number n of voters, anonymous, sovereign and strategy-proof social choice functions if and only if X is not blocked.

Theorem

The generalized single-peaked domain $\mathcal{S}(X)$ admits sovereign and strategy-proof social choice functions that are neutral with an odd number of individuals [and anonymous] if and only if X is a median space.