Basics of Coopertive Games

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I. Cooperative Games: Transferable Utility framework

A *cooperative game* is a pair (N, v)
$N = \{1, ..., n\}$ is the set of agents/players; $S \subset N$ are called "coalitions"
$v : 2^N \to \mathbb{R}_+$ is the "characteristic (value) function"
$v(S)$ is the total value for coalition S
Assumptions: $v(\emptyset) = 0$; *monotonicity* $S \subset N \Longrightarrow v(S) \leq v(N)$

Utilities are "transferable". Benchmark situation: players' utility is quasi-linear in money and side payments are feasible. When the outcome of the game is a, and the money player i gets is t_i, her utility is $u_i(a) + t_i$.

Allocation to N: $x = (x_1, ..., x_n) \in \mathbb{R}^n$
Feasible? Efficient? Stable (agents can agree on it)?

Two dual interpretations for $v(S)$:
surplus sharing game, $v(S)$ is the (largest) surplus coalition S can get
cost sharing game, $v(S) = c(S)$ is the (smallest) cost to serve coalition S

Equivalence:

For each cost game (N,c), the associated surplus game (N,v) is one with $v(S) = \sum_{i \in S} c(\{i\}) - c(S)$ for all $S \subset N$. Here $v(S)$ is interpreted as the surplus agents from S can get if they join together, compared to being served separately.

Example:

$N = \{A, B, C\}$ - three towns, need water supply system.
Cost: $c(A) = 120$, $c(B) = 140$, $c(C) = 120$, $c(A,B) = 170$, $c(B,C) = 190$, $c(A,C) = 160$, $c(A,B,C) = 255$.
Associated surplus sharing game: $v(A) = v(B) = v(C) = 0$, $v(A,B) = 90$, $v(B,C) = 70$, $v(A,C) = 80$, $v(A,B,C) = 125$.
The only efficient way is to build the system together.

Notation:

x_S is the $|S|$-dimensional restriction of $x = (x_1, ..., x_n)$ to only those its coordinates which belong to S. We will also use $x(S) = \sum_{i \in S} x_i$.

A partition of N is a collection $\pi = \{S_1, ..., S_k\}$ of subsets of N, such that they do not intersect ($S_i \cap S_j = \emptyset$) and their union is the whole N ($\bigcup_{1 \le i \le k} S_i = N$).

If a partition $\pi = \{S_1, ..., S_k\}$ forms, then each coalition $S_i \in \pi$ can obtain $v(S_i)$, which (given TU assumption) can be divided between its members in an arbitrary way.
Potentially, a coalition could also waste some of its generated utility, or share it with other agents, who do not belong to this coalition.

Feasibility:
A vector $x = (x_1, ..., x_n)$ is feasible if and only if there exists a partition $\pi = \{S_1, ..., S_k\}$ of our society, such that $x_1 + ... + x_n \leq v(S_1) + ... + v(S_k)$ (for cost sharing: $x_1 + ... + x_n \geq c(S_1) + ... + c(S_k)$).

Efficiency:
$x_1 + ... + x_n = M = \max_\pi (v(S_1) + ... + v(S_k))$, where maximum is taken over all partitions $\pi = \{S_1, ..., S_k\}$ of N.
UPS (utility possibility set) is a half-space, and has the utility possibility frontier (UPF) which is a hyperplane, orthogonal to the vector $(1, ..., 1)$. I.e., UPS is the set of all $(u_1, ..., u_n)$ such that $u_1 + ... + u_n \leq M$ ($\geq$ for cost sharing).

Individual rationality:
Surplus sharing: $x_i \geq v(\{i\})$; cost-sharing: $x_i \leq c(\{i\})$.

Efficient and individually rational feasible allocations in a given TU game are called **imputations**.

Usual assumptions:

super(sub)-additivity: $S \cap S' = \varnothing \Rightarrow v(S) + v(S') \leq v(S \cup S')$ (resp. $\geq$)
super(sub)-modularity: $v(S) + v(S') \leq v(S \cup S') + v(S \cap S')$ (resp. $\geq$)
Note: "super" is for profit sharing and "sub" for cost sharing games

For super(sub)-additive games, the only efficient outcomes are those when "grand coalition" N forms Thus, the only remaining question is: how will, or should, the agents divide $v(N)$ (or $c(N)$), taking into account the whole characteristic function.

A feasible and efficient surplus (cost) allocations are $(x_1, ..., x_n)$, such that $x_1 + x_2 + ... + x_n = v(N)$ (or $c(N)$).

In a super(sub)-additive game the set of imputations is non-empty:
$x_j = v(\{j\}) + \frac{1}{n}\left(v(N) - \sum_{i \in i} v(\{j\})\right)$ for all $j \in N$ is an imputation.

Stand alone principle (for the distribution of total surplus/cost):
In surplus sharing games: $x(S) = \sum_{i \in S} x_i \geq v(S)$ for any $S \subset N$.
In cost sharing games: $x(S) = \sum_{i \in S} x_i \leq c(S)$ for any $S \subset N$.
<u>Note:</u> if a (feasible) allocation satisfies stand alone principle, then it is efficient (Pareto efficiency is just the stand alone requirement for N).

The **core** is the set of feasible allocations satisfying the stand alone principle.

<u>Alternative definition:</u>
Coalition S **blocks** an allocation $x \in \mathbb{R}^n$, if $v(S) > \sum_{j \in S} x_j$
($c(S) < \sum_{j \in S} x_j$ for cost sharing).
The **core** is the set of feasible allocations which cannot be blocked.

The no-subsidy principle:

In cost sharing games: $x(S) = \sum_{i \in S} x_i \geq c(N) - c(N \backslash S)$

In surplus sharing games: $x(S) = \sum_{i \in S} x_i \leq v(N) - v(N \backslash S)$

Note: on the set of efficient allocations, ones with $\sum x_i = c(N)$ (or $v(N)$), the two are equivalent: $x(S) = \sum_{i \in S} x_i \geq c(N) - c(N \backslash S) = \sum_{i \in N} x_i - c(N \backslash S)$ is equivalent to $\sum_{i \in N \backslash S} x_i \leq c(N \backslash S)$.

Important: the core of a super(sub)-additive game may be empty, or it may be quite large.

Example: $N = \{1, 2, 3\}$, $v(i) = 0$, $v(N) = 1$, If $v(i, j) = \frac{3}{4}$, core is empty. If $v(i, j) = \frac{1}{4}$, core is all x such that $0 \leq x_i \leq \frac{3}{4}$, and $x_1 + x_2 + x_3 = 1$

Values of cooperative games:

"Value" is a rule φ which, for each game (N, v) specifies a unique feasible allocation $\varphi(N, v) = x \in \mathbb{R}^n$.

Desirable properties:
– Efficiency (we include it in the definition of value)
– Individual Rationality (IR): $\varphi_i(N, v) \geq v(\{i\})$ for all $i \in N$
– Core Selection (CS): $\varphi(N, v) \in$ core(N, v) whenever the core is non-empty
– Coalitional Monotonicity (CM): $\{v(S_0) < v'(S_0)$ and $v(S) = v'(S)$ for all $S \neq S_0\} \Rightarrow x_i \leq x_i'$ for all $i \in S_0$

Proposition: *CS and CM are not compatible*

(proof by a simple counterexample with five agents)

The *marginal contribution* allocation for an ordering $i_1, i_2, \cdots$, of the agents: $x_{i_1} = v(\{i_1\}), x_{i_2} = v(\{i_1, i_2\} - v(\{i_1\}), x_{i_3} = v(\{i_1, i_2, i_3\}) - v(\{i_1, i_2\}), \cdots$

Shapley value $Sh(N, v)$ is the average of all marginal contribution vectors

$$Sh_i(N, v) = x_i = \sum_{0 \le s \le n-1} \frac{s!(n-s-1)!}{n!} \sum_{S \subseteq N \setminus i, |S| = s} [v(S \cup \{i\}) - v(S)]$$

Shapley characterized his solution by the combination of Anonymity, Additivity, and the Dummy axioms:

Anonymity: fix $i, j \in N$; if $\forall S \subset N \backslash \{i, j\}$ we have $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\varphi_i = \varphi_j$

Dummy: if $v(S \cup \{i\}) = v(S)$ for any $\forall S \subset N \backslash \{i\}$, then $\varphi_i = 0$

Additivity: $\varphi(N, v + w) = \varphi(N, v) + \varphi(N, w)$

Many alternative characterizations followed, vindicating the star status of this solution for TU cooperative games.

Shapley value is Individually Rational (for super(sub)-additive games), and Coalition Monotonic, but does not satisfy core selection. However:

Proposition: *if (N, v) is super(sub)-modular, the core is the convex hull of the marginal contribution vectors (a characteristic property of super(sub)-modularity)*

Corollary: *if (N, v) is super(sub)-modular the Shapley value is the "center" of the Core*

Definition $\forall x \in \mathbb{R}^n$ let x^* be the vector obtained by re-ordering coordinates of x in increasing order

<u>Lexicographic Ordering:</u> $\forall x, y \in \mathbb{R}^n$, $x \succeq_{\text{lex}} y$ iff $\exists k$ such that $x_i^* = y_i^*$ for $1 \leq i \leq k$, and $x_{k+1}^* > y_{k+1}^*$

<u>Lorenz Dominance:</u> $x \succeq_{\text{lor}} y$ iff $\sum_{1 \leq i \leq k} x_i^* \geq \sum_{1 \leq i \leq k} y_i^*$ for all k, $1 \leq k \leq n$

Proposition: *if v is super(sub)-modular, the core admits a Lorenz dominant selection, called the Egalitarian (or Dutta Ray) solution.*

Fix a game (N, v) and a solution φ defined for all its *subgames* (S, v^S): $S \subseteq N$ and $v^S(T) = v(T)$ for all $T \subseteq S$

Population Monotonicity (PM^+) (resp. PM^-):

$\{S \subseteq N \diagdown j$ and $i \in S\} \Rightarrow \varphi_i(S, v^S) \leq \varphi_i(S \cup \{j\}, v^{S \cup \{j\}})$ (resp. $\geq$)

PM^+ means that adding a new agent increases (weakly) the shares of existing ones; PM^- means it decreases them (weakly).
Both properties are of central interest in cost sharing

Proposition: *the Shapley value as well as the Egalitarian (Dutta Ray) solution are* PM^+ *(resp.* PM^-*) in supermodular (resp. submodular) games*

Nucleolus

Given (N, v) , $S \subset N$, and an allocation $x \in \mathbb{R}^n$, we define excess $e(S, x)$ of coalition S at allocation x to be
$e(S, x) = v(S) - x(S) = v(S) - \sum_{i \in S} x_i$.

We can interpret a positive excess ($e(S, x) > 0$) as the amount of dissatisfaction or complaint of the members of S at the allocation x.
We can use the excess to define the core:
core$(N, v) = \{x \in \mathbb{R}^n : x$ is an imputation and $\forall S \subseteq N,\ e(S, x) \leq 0\}$

Nucleolus: $Nu(N, v)$ is the set of imputations z which lexicographically maximize the vector $(-e(S, z))_{S \subset N} \in \mathbb{R}^{2^n}$.
I.e., $Nu(N, v)$ is the set of such imputations z that $(-e(S, z))_{S \subset N} \succeq_{\text{lex}} (-e(S, y))_{S \subset N}$ for any other imputation y in the game (N, v).

Thus, Nucleolus aims to minimize the dissatisfaction of the coalition with the largest complaint (and then that of the 2-nd worst off coalition, etc.)

Fact: any compact set $C \subset \mathbb{R}^n$ has a maximum with respect to $\succeq_{\text{lex}}$, and if C is convex this maximum is unique.

For super(sub)-additive games, the set of imputations is non-empty, compact, and convex. Given this, the set $\{(-e(S,x))_{S\subset N} : x \text{ is an imputation}\}$ can be shown to also be non-empty, compact, and convex.
Hence, the Nucleolus is non-empty, and (because of convexity) is a singleton.
Thus:

Proposition: *Nucleolus is a value for super(sub)-additive games.*

Nucleolus satisfies Individual Rationality and Core Selection (if the core is non-empty, it is in the core).
It satisfies Anonymity and Dummy axioms (but not Additivity).

II. Bargaining

Let $N = \{1, ..., n\}$. Each agent has "reservation utility" d_i which she can guarantee herself no matter what the other agents do. It is often called "outside option". Together they can reach any joint outcome $(u_1, u_2, ..., u_n) \in S \subset \mathbb{R}^n$. This S is the utility possibility set (UPS). We assume
1) Normalization: "disagreement point" $d = (d_1, ..., d_n) = \overline{0}$.
2) S is convex, comprehensive, $S \cap \mathbb{R}^n_+$ is compact; $\exists u \in S : u \gg \overline{0}$;
we may assume $S \subset \mathbb{R}^n_+$ (and talk of comprehensiveness restricted to $\mathbb{R}^n_+$)
3)* "Minimal Transferability": for all $u \in S$ and all $i \in N$ we have that, whenever $u_i > 0$, there exists $v \in S$ with $v_i < u_i$ and $v_j > u_j$ for all $j \neq i$.

A **bargaining problem** is (S, d) (under normalization $d = \overline{0}$ just D)

A **bargaining solution** for a domain D of bargaining problems: $\psi : D \rightarrow U$, which associates to each $(U, d) \in D$ an allocation $u = \psi((U, d)) \in U$.

Properties: Individual Rationality: $\psi_i \geq d_i$; Efficiency: $\psi \in$UPF

Some well-known solutions (assume $d = \bar{0}$, $S \subset \mathbb{R}^n_+$):

Notation: $\forall x, y \in \mathbb{R}^n$, we define:
$x \geq y \overset{\text{def}}{\Longleftrightarrow} \forall i \in N : x_i \geq y_i$;
$x > y \overset{\text{def}}{\Longleftrightarrow} \forall i \in N : x_i \geq y_i$, and $\exists j \in N : x_j > y_j$;
$x \gg y \overset{\text{def}}{\Longleftrightarrow} \forall i \in N : x_i > y_i$

Nash Solution: $N(S) = \arg\max_{x \in S} \prod_{1 \leq i \leq n} x_i$

Kalai-Smorodinsky Solution: let $m \in \mathbb{R}^n_+$ be defined by $m_i = \max\{x_i : x \in S\}$, $[\bar{0}, m] = \{x \in \mathbb{R}^n_+ : x = \lambda\bar{0} + (1-\lambda)m, \lambda \in [0,1]\}$; $K(S) = \max[\bar{0}, m] \cap S$

Egalitarian Solution: $\max\{x \in S : \exists c \; \forall i \; x_i = c\}$

Axioms:

Symmetry: if S is invariant under all exchanges of agents, then $\psi_i(S) = \psi_j(S)$ $\forall i, j \in N$

Let $\pi : N \longrightarrow N$ be a bijection; define $\widetilde{\pi} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ to be $\widetilde{\pi}(x) = (x_{\pi(1)}, ..., x_{\pi(n)})$ for any $x = (x_1, ..., x_n) \in \mathbb{R}^n$.
Anonymity: $\psi(\widetilde{\pi}(S)) = \widetilde{\pi}(\psi(S))$ (more general then symmetry)

Scale Invariance: $\psi((\lambda S)) = \lambda\psi(S)$ for any $\lambda \in \mathbb{R}^n_{++}$
(here $\lambda\Omega = \{\lambda \cdot x : x \in \Omega\}$, for any $\Omega \subset \mathbb{R}^n$)

Contraction Independence: if $S' \subset S$ and $\psi(S) \in S'$, then $\psi(S') = \psi(S)$

Proposition: *Nash solution is the only one satisfying Symmetry, Scale Invariance, and Contraction Independence*
Note: if we do not require Efficiency, the only other solution is $\psi(S) \equiv \overline{0}$ for all S